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LETTER TO THE EDITOR

A new boson realization of the su(3) algebra

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Abstract. A new boson realization of su(3) has been given and a set of recursion formulae of the transformation matrix, which turns the Dyson into the Holstein–Primakoff representation, has been presented. The multiplicity that may occur in the reduction of $su(3) \supset u(1)+u(1)$ is automatically determined.

In this letter, a new boson representation will be given for the generators of the su(3) algebra in the Cartan standard basis $su(3) \supset u(1)+u(1)$ by means of the generalized coherent state (GCS) technique. The GCS theory has been widely discussed in the literature (Dobaczewski 1981, 1982, Rowe 1985, Hecht 1987, Zhang *et al* 1989). A main problem of this theory is to find its measure and to calculate the necessary integrals. For most of the higher symmetry algebras, however, even if the measure can be found, it is often so complicated that the necessary integrals become very cumbersome and difficult to use in practice. A basic idea for overcoming this difficulty is to change the measure, turning the GCS onto the Bargmann measure (Dobaczewski 1981, 1982, Rowe 1985). The price paid for it is that the unitarity of the operators is partly lost. The second step is to find a similarity transformation such that the transformed operators restore unitarity.

From definition, we know that the GCS depends sensitively on the coset space of a group G, i.e. an irreducible representation (irrep) of G. For the SU(3) group, the non-normalized GCS can be written as

$$|\Lambda\{C\}\rangle = \exp\left[\sum_{\Lambda\cdot\delta>0} C_\delta E_{-\delta}\right]|\phi_0\rangle \tag{1}$$

where $\{H_1, H_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm(\alpha+\beta)}\}$ are the generators of the su(3) algebra in the Cartan standard basis; α and β are the two simple roots, the Dynkin diagram is as follows

$$\begin{array}{ccc} \lambda & & \mu \\ \circ & \text{———————} & \circ \\ \alpha & & \beta \end{array}$$

here λ and μ are the two non-negative integers, and (λ, μ) label the irreps of the su(3) algebra; $\{C_\delta\}$ are complex variables which will be equivalently numbered by the root vector themselves; $|\phi_0\rangle$ is the highest-weight state with highest weight Λ. Because

$$\Lambda = \frac{2\lambda+\mu}{3}\alpha + \frac{\lambda+2\mu}{3}\beta \qquad \alpha, \beta \in \text{simple roots} \tag{2}$$

therefore $\Lambda\cdot\alpha = \lambda/6$, $\Lambda\cdot\beta = \mu/6$. Obviously, when $\lambda = 0$ (or $\mu = 0$) the maximal stability subalgebra is u(2) with $\{H_1, H_2, E_\alpha, E_{-\alpha}\}$ (or $\{H_1, H_2, E_\beta, E_{-\beta}\}$). Thus the coset space

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is SU(3)/U(2). On the other hand for arbitrary irreps of the SU(3) group, i.e. $\lambda \neq 0$ and $\mu \neq 0$. The maximal stability subalgebra is $U(1) \times U(1)$ with the generators $\{H_1, H_2\}$, therefore the corresponding coset space is not SU(3)/U(2) but $SU(3)/U(1) \times U(1)$. For the latter, if we attempt to construct the GCS using the subgroup U(2), then the so-called vector coherent state would be obtained (Hecht 1987).

For the sake of simplicity, we redefine

$$\begin{aligned} &A_1 = \sqrt{24}\, E_\alpha && A_2 = \sqrt{24}\, E_\beta && A_3 = \sqrt{24}\, E_{\alpha+\beta} \\ &B_1 = \sqrt{24}\, E_{-\alpha} && B_2 = \sqrt{24}\, E_{-\beta} && B_3 = \sqrt{24}\, E_{-\alpha-\beta} \\ &C_1 = \sqrt{12}\, H_1 && C_2 = 6H_2 \end{aligned} \tag{3}$$

such that the canonical commutation relations (Wybourne 1974) of the generators in the Cartan basis become

$$[C_1, C_2] = 0 \tag{4a}$$

$$\begin{aligned} &[C_1, A_1] = A_1 && [C_1, A_2] = A_2 && [C_1, A_3] = 2A_3 \\ &[C_2, A_1] = 3A_1 && [C_2, A_2] = -3A_2 && [C_2, A_3] = 0 \end{aligned} \tag{4b}$$

$$\begin{aligned} &[A_1, A_2] = 2A_3 && [A_1, A_3] = 0 && [A_2, A_3] = 0 \\ &[A_1, B_1] = 2(C_1 + C_2) && [A_1, B_2] = 0 && [A_1, B_3] = -2B_2 \end{aligned} \tag{4c}$$

$$[A_2, B_2] = 2(C_1 - C_2) \qquad [A_2, B_3] = 2B_1 \qquad [A_3, B_3] = 4C_1. \tag{4d}$$

At present, the GCS of the group chain $SU(3) \supset U(1) \times U(1)$ can be rewritten as

$$|Z\rangle = \exp\{Z_1^* B_1 + Z_2^* B_2 + Z_3^* B_3\} |\phi_0\rangle \tag{5}$$

where $|\phi_0\rangle = |^{(\lambda,\mu)}_{\lambda+\mu,\lambda-\mu}\rangle$, $\lambda + \mu$ and $\lambda - \mu$ are the eigenvalues of the operators C_1 and C_2, respectively.

As mentioned above, the next step is to introduce the Bargmann space. It is well known that the Bargmann space is isomorphic with the many-boson space, so the states and operators of an algebra may directly be turned onto the boson space by means of an Usui-like operator (Dobaczewski 1981, 1982).

Here, the Usui-like operator is defined by

$$|\psi)_B = V|\psi\rangle \tag{6}$$

$$V = \langle\phi_0| \exp\{b_1^+ A_1 + b_2^+ A_2 + b_3^+ A_3\}|0) \tag{7}$$

where $|\psi)_B$ is called the boson image of the state $|\psi\rangle$, for example the boson vacuum $|0)$ is the image of the highest weight state $|\phi_0\rangle$. Note that the space defined by (6) and (7) is a subspace of the many-boson space, and called the physical subspace by physicists. Corresponding to the transformation equations (6) and (7), the boson image $O^{(D)}$ of an arbitrary operator $\hat{O}$ of the su(3) algebra can determined by

$$|\phi\rangle = \hat{O}|\psi\rangle \to |\phi)_B = O^{(D)}|\psi)_B \tag{8}$$

where the $O^{(D)}$ operator is defined by

$$V\hat{O}\bar{V} = O^{(D)} V\bar{V} \tag{9}$$

and $\bar{V}$ is determined by $\bar{V}V = 1$, $V\bar{V} \neq 1$.

From (9), obviously we have

$$O^{(\mathrm{D})}V\bar{V}=V\hat{O}\bar{V}=\langle\phi_0|\left\{O+\sum_m\frac{1}{m!}[X,[X,\ldots,[X,O]\ldots]\,\mathrm{e}^X\right\}|0)\,\bar{V} \tag{10}$$

where $X=b_1^+A_1+b_2^+A_2+b_3^+A_3$. Using (10), we immediately arrive at the following results:

$$A_1\to A_1^{(\mathrm{D})}=b_1-b_2^+b_3 \qquad A_2\to A_2^{(\mathrm{D})}=b_2+b_1^+b_3 \qquad A_3\to A_3^{(\mathrm{D})}=b_3$$

$$B_1\to B_1^{(\mathrm{D})}=4\lambda b_1^+-4b_1^+b_1^+b_1-2(b_3^+-b_1^+b_2^+)b_2-2(b_3^++b_1^+b_2^+)b_1^+b_3$$

$$B_2\to B_2^{(\mathrm{D})}=4\mu b_2^+-4b_2^+b_2^+b_2+2(b_3^++b_1^+b_2^+)b_1-2(b_3^+-b_1^+b_3^+)b_2^+b_3$$

$$\begin{aligned}B_3\to B_3^{(\mathrm{D})}=&\,4(\lambda+\mu)b_3^++4(\lambda-\mu)b_1^+b_2^+-4b_3^+(b_1^+b_1+b_2^+b_2)-4b_1^+b_2^+(b_1^+b_1-b_2^+b_2)\\&-4(b_3^+b_3^++b_1^+b_1^+b_2^+b_2^+)b_3\end{aligned}$$

and

$$\begin{aligned}&C_1\to C_1^{(\mathrm{D})}=\lambda+\mu-b_1^+b_1-b_2^+b_2-2b_3^+b_3\\&C_2\to C_2^{(\mathrm{D})}=\lambda-\mu-3b_1^+b_1+3b_2^+b_2.\end{aligned} \tag{11}$$

Operators $A_i^{(\mathrm{D})}$, $B_i^{(\mathrm{D})}$, $C_i^{(\mathrm{D})}$ are considered as generalized Dyson realizations of A_i, B_i, C_i su(3), $i=1, 2, 3$. Obviously, the operators of (11) are partly non-unitary. The origin of this is due to the non-orthonormality of the physical boson basis vectors. It can be restored by introducing a new map

$$|\psi_{\alpha\beta}^{(\lambda\mu)\gamma})_{\mathrm{B}}=K|Z_{\alpha\beta\gamma}) \tag{12}$$

where $|Z_{\alpha\beta\gamma})$ are a set of orthonormalized basis vectors of the boson space

$$|Z_{\alpha\beta\gamma})=\frac{(b_1^+)^{\alpha/2+\beta/6-\gamma}(b_2^+)^{\alpha/2-\beta/6-\gamma}(b_3^+)^{\gamma}|0)}{\sqrt{(\alpha/2+\beta/6-\gamma)!(\alpha/2-\beta/6-\gamma)!\gamma!}} \tag{13}$$

where $b_i^+(b_i)$, $i=1, 2, 3$, are the ideal boson creation (annihilation) operators; the quantum numbers α, β and γ satisfy the following selection rules:

$$\begin{aligned}&\alpha, \beta \text{ and } \gamma=0 \text{ or positive integers}\\&\alpha/2\pm\beta/6-\gamma=0 \text{ or positive integers}\end{aligned} \tag{14a}$$

$$\begin{aligned}&\alpha=0,1,2,\ldots,2(\lambda+\mu)\\&\beta=\pm3\alpha,\pm3(\alpha-2),\pm3(\alpha-4),\ldots,\begin{cases}0 & \text{if } \alpha=\text{even}\\ \pm3 & \text{if } \alpha=\text{odd}\end{cases}\\&\gamma=0,1,2,\ldots,(\alpha/2-|\beta|/6).\end{aligned} \tag{14b}$$

Corresponding to the transformation (12), we have a similarity transformation

$$O^{(\mathrm{HP})}=K^{-1}O^{(\mathrm{D})}K. \tag{15}$$

Because, by definition,

$$(O^{(\mathrm{HP})})^+ = (O^+)^{(\mathrm{HP})} \tag{16}$$

is valid, hence K must satisfy

$$KK^+(O^{(\mathrm{D})}) = (O^+)^{(\mathrm{D})}KK^+. \tag{17}$$

Equation (17) can be solved for KK^+. The $O^{(\mathrm{HP})}$ is called the Holstein-Primakoff realization of the operator $\hat{O}$. Because $(C_i^+)^{(\mathrm{D})} = (C_i^{(\mathrm{D})})^+$ and $(C_i^{(\mathrm{D})})^+ = C_i^{(\mathrm{D})}$, $i = 1, 2$, therefore

$$[C^{(\mathrm{D})}, KK^+] = 0 \qquad i = 1, 2. \tag{18}$$

This means that KK has been diagonalized with respect to the quantum numbers α and β.

Using the basis (13), we can obtain a set of equations for KK^+ as follows:

$$\begin{aligned}
&(4\lambda - \alpha - \beta)\sqrt{\alpha/2+\beta/6-\gamma'+1}(KK^+)_{\gamma',\gamma}(\alpha, \beta) - 2\sqrt{\gamma'(\alpha/2-\beta/6-\gamma'+1)}\\
&\qquad \times (KK^+)_{\gamma'-1,\gamma}(\alpha, \beta)\\
&\qquad -2\sqrt{(\gamma'+1)(\alpha/2+\beta/6-\gamma')(\alpha/2+\beta/6-\gamma'+1)(\alpha/2-\beta/6-\gamma')}\\
&\qquad \times (KK^+)_{\gamma'+1,\gamma}(\alpha, \beta)\\
&\quad = \sqrt{\alpha/+\beta/6-\gamma+1}(KK^+)_{\gamma',\gamma}(\alpha', \beta')\delta_{\alpha',\alpha+1}\delta_{\beta',\beta+3}\\
&\qquad -\sqrt{(\gamma+1)(\alpha/2-\beta/6-\gamma)}(KK^+)_{\gamma',\gamma+1}(\alpha', \beta')\delta_{\alpha',\alpha+1}\delta_{\beta',\beta+3}
\end{aligned} \tag{19a}$$

$$\begin{aligned}
&(4\mu - \alpha + \beta)\sqrt{\alpha/2-\beta/6-\gamma'+1}(KK^+)_{\gamma',\gamma}(\alpha, \beta) + 2\sqrt{\gamma'(\alpha/2+\beta/6-\gamma'+1)}\\
&\qquad \times (KK^+)_{\gamma'-1,\gamma}(\alpha, \beta)\\
&\qquad +2\sqrt{(\gamma'+1)(\alpha/2+\beta/6-\gamma')(\alpha/2-\beta/6-\gamma'+1)(\alpha/2-\beta/6-\gamma')}\\
&\qquad \times (KK^+)_{\gamma'+1,\gamma}(\alpha, \beta)\\
&\quad = \sqrt{\alpha/-\beta/6-\gamma+1}(KK^+)_{\gamma',\gamma}(\alpha', \beta')\delta_{\alpha',\alpha+1}\delta_{\beta',\beta-3}\\
&\qquad -\sqrt{(\gamma+1)(\alpha/2+\beta/6-\gamma)}(KK^+)_{\gamma',\gamma+1}(\alpha', \beta')\delta_{\alpha',\alpha+1}\delta_{\beta',\beta-3}
\end{aligned} \tag{19b}$$

$$\begin{aligned}
&4(\lambda - \mu - \beta/3)\sqrt{(\alpha/2+\beta/6-\gamma'+1)(\alpha/2-\beta/6-\gamma'+1)}(KK^+)_{\gamma',\gamma}(\alpha, \beta)\\
&\qquad +4(\lambda+\mu-\alpha+\gamma'-1)\sqrt{\gamma'}(KK^+)_{\gamma'-1,\gamma}(\alpha, \beta)\\
&\qquad -4\sqrt{\begin{aligned}&(\gamma'+1)(\alpha/2+\beta/6-\gamma'+1)(\alpha/2+\beta/6-\gamma')\\ &\quad\times(\alpha/2-\beta/6-\gamma'+1)(\alpha/2-\beta/6-\gamma')\end{aligned}}\\
&\qquad \times (KK^+)_{\gamma'+1,\gamma}(\alpha, \beta)\\
&\quad = \sqrt{\gamma+1}(KK^+)_{\gamma'\gamma+1}(\alpha', \beta')\delta_{\alpha',\alpha+2}\delta_{\beta'\beta}.
\end{aligned} \tag{19c}$$

For instance for (19a)

$$\alpha = 0 \qquad \beta = 0 \qquad \gamma = 0$$

$$(KK^+)_{00}(0, 0) = 1$$

and for (19*b*)

$$\alpha=1 \qquad \beta=\pm 3 \qquad =0$$

$$(KK^{+})_{00}(1,3)=4\lambda$$

$$(KK^{+})_{00}(1,-3)=4\mu$$

$$\alpha=2 \qquad \beta=\pm 6 \qquad \gamma=0$$

$$(KK^{+})_{00}(2,6)=16\lambda(\lambda-1)$$

$$(Kk^{+})_{00}(2,-6)=16\mu(\mu-1)$$

$$\alpha=2 \qquad \beta=0 \qquad \gamma=0,1$$

$$(KK^{+})(2,0)=\begin{pmatrix}16\lambda\mu-4\lambda+4\mu & 4\lambda-4\mu\\ 4\lambda-4\mu & 4\lambda+4\mu\end{pmatrix}$$

$$\alpha=3 \qquad \beta=\pm 9 \qquad \gamma=0$$

$$(KK^{+})_{00}(3,9)=64\lambda(\lambda-1)(\lambda-2)$$

$$(KK^{+})_{00}(3,-9)=64\mu(\mu-1)(\mu-2)$$

$$\alpha=3 \qquad \beta=\pm 3 \qquad \gamma=0,1$$

$$(KK^{+})(3,3)=\begin{pmatrix}32\lambda(2\lambda\mu+\lambda-\mu-1) & 16\sqrt{2}\lambda(\lambda-\mu-1)\\ 16\sqrt{2}\lambda(\lambda-\mu-1) & 16\lambda(\lambda+\mu-1))\end{pmatrix}$$

$$(Kk^{+})(3,-3)=\begin{pmatrix}32\mu(2\lambda\mu+\mu-\lambda-1) & 16\sqrt{2}\mu(\lambda-\mu+1)\\ 16\sqrt{2}\mu(\lambda-\mu+1) & 16\mu(\lambda+\mu-1)\end{pmatrix}.$$

Because the operator KK^{+} is Hermitian, we can diagonalize it by means of a unitary matrix U,

$$U^{-1}KK^{+}U=\begin{pmatrix}a_1 & & \\ & a_2 & \\ & & \ddots\end{pmatrix}. \tag{20}$$

The eigenvalues $a_1, a_2, \ldots,$ are real numbers. From (20) we immediately arrive at K and its inverse K^{-1}

$$K=U\begin{pmatrix}\sqrt{a_1} & & \\ & \sqrt{a_2} & \\ & & \ddots\end{pmatrix}U^{-1} \qquad K^{-1}=U\begin{pmatrix}1/\sqrt{a_1} & & \\ & 1/\sqrt{a_2} & \\ & & \ddots\end{pmatrix}U^{-1}. \tag{21}$$

Having found K and K^{-1}, using (12) and (13), the basis vectors of the representation space can be given.

So far, we have explicitly given a set of recursion formulae for the transformation matrix for the standard reduction of $su(3)\supset u(1)+u(1)$, which is a transformation from a non-unitary Dyson to a unitary Holstein–Primakoff realization. Another remarkable result is that it naturally solves the multiplicity problem that can occur in the reduction of $su(3)\supset u(1)+u(1)$, and the multiplicity is just equal to the dimension of the K matrix.

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